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A COUNTEREXAMPLE ON EXTREME OPERATORS†

BY

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ABSTRACT

We give examples of extreme operators in the unit ball of $L(C(X), C(Y))$ which are not nice operators (i.e., their adjoints do not carry extreme points into extreme points in the corresponding unit balls.) Such counterexamples exist in fact for every non-dispersed compact Hausdorff space X when the scalars are complex.

1. Introduction

Let X, Y be compact Hausdorff spaces. $C(X)$ (resp. $C(Y)$) will denote the Banach space of continuous functions on X (resp. Y) with the sup-norm. $L(C(X), C(Y))$ is the space of bounded linear operators from $C(X)$ into $C(Y)$. Our problem is to characterize the extreme points in the unit ball of $L(C(X), C(Y))$, called in short *extreme operators*. To each T in $L(C(X), C(Y))$ there corresponds a w^*-continuous adjoint mapping $T^*: Y \to C(X)^*$ such that $T^*y(f) = Tf(y)$ for each $f \in C(X), y \in Y$ (cf. [4, p. 490]). T is extreme if and only if the only w^*-continuous $U^*: Y \to C(X)^*$ such that $\| T^*y \pm U^*y \| \leqq 1$ for each $y \in Y$, is the identically zero mapping. We say that T in $L(C(X), C(Y))$ is a *nice operator* if T^*y is extreme in the unit ball of $C(X)^*$ for each $y \in Y$. In other words, T^* maps extreme points to extreme points in the corresponding unit balls. Obviously, all nice operators are extreme, and the characterization problem is actually: must every extreme operator in $L(C(X), C(Y))$ be nice? This problem was posed in 1965 by Blumenthal, Lindenstrauss and Phelps [2] and since then has been answered in the affirmative only in some special cases. We list here the most important cases:

a) Real scalars, X is metric [2].

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b) Real scalars, X is Eberlein-compact, Y is metric [1].
c) Real scalars, T is weakly compact, Y is separable [3].
d) Real or complex scalars, T is compact [2], [8].
e) Real or complex scalars, X is dispersed [9].
f) Real or complex scalars, Y is extremally-disconnected [9].

In this paper we show that if the scalars are complex, then in general the answer is negative, and for any non-dispersed compact Hausdorff space X, there is a compact Hausdorff space Y and an extreme operator T in $L(C(X), C(Y))$ that is not nice. These counterexamples are surprisingly simple. For example, if X is perfect and S_X is the unit ball of $C(X)^*$ with the w^*-topology, then $T: C(X) \to C(S_X)$ such that T^* is the identity on S_X is the required counterexample. In the real case, the situation is much more complicated, but I have recently shown that there exist non-nice extreme operators in the real case too (*A non-nice extreme operator*, to appear in Israel J. Math.). In fact, the same T as above is extreme and not nice if X is a certain non-metrizable Eberlein compact space.

Our notations are fairly standard. $C(X)^*$ is identified with the space of all Borel countably-additive regular measures on X. δ_x denotes the evaluation-measure at $x \in X$. If $\mu \in C(X)^*$ and f is any μ-integrable function, then $\mu(f)$ will denote $\int_X f d\mu$, and $f \cdot \mu$ will denote their Radon–Nykodim multiplication; that is, $\nu = f \cdot \mu$ is the measure on X such that $\nu(E) = \int_E f d\mu$ for each Borel-set $E \subset X$. We denote the unit ball of $C(X)^*$ by S_X.

Our first result is a property of w^*-continuous mappings $F: S_X \to S_X$ such that $\|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in S_X$ (Theorem 2.3). We use this property to obtain our main result — the existence of extreme non-nice operators in the complex case (Theorem 2.5, Corollary 2.6 and Theorem 2.8).

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2. The counterexamples

Let X be a compact Hausdorff space. Let S_X be the unit ball of $C(X)^*$, equipped with the w^*-topology. Our first objective is to investigate some properties of continuous mappings $F: S_X \to S_X$ such that $\|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in S_X$. The scalars are either real or complex, unless otherwise specified.

We begin with a probably known lemma, but we haven't found the proof in the literature. Hustad [5, lemma 3.2] proves an analogous result for l^1_n.

LEMMA 2.1. *Let $\mu, \nu \in S_X$ such that $\|\mu\| = 1$ and $\|\mu \pm \nu\| \leqq 1$. Then there* *exists a* real *Borel-measurable function f such that* $-1 \leqq f \leqq 1$, $|\mu|(f) = 0$, *and* $\nu = f \cdot \mu$.

Proof. We can write $\nu = \nu_1 + \nu_2$, where $\nu_1, \nu_2 \in S_X$, ν_1 is absolutely continuous with respect to μ, and ν_2 is singular to μ. Hence

$$1 = \| \mu \pm \nu \| = \| \mu \pm (\nu_1 + \nu_2) \| = \| \mu \pm \nu_1 \| + \| \nu_2 \|$$

and so $\nu_2 = 0$ and ν is absolutely continuous with respect to μ. So, by the Radon–Nykodim theorem, $\nu = f \cdot \mu$ for some $f \in L^1(\mu)$. Let us write $f = f_1 + if_2$ for real f_1, f_2 (in the real case, $f_2 \equiv 0$ of course). Then

$$1 \geqq \| \mu \pm \nu \| = \int_X |1 \pm (f_1 + if_2)| \, d|\mu| \geqq \int_X |1 \pm f_1| \, d|\mu|$$

and so

$$1 = \| \mu \| \leqq \int_X \left\{ \frac{1}{2}|1 + f_1| + \frac{1}{2}|1 - f_1| \right\} d|\mu| \leqq 1.$$

Hence we must have equality everywhere and so $f_2 = 0$ μ-a.e. and we may assume therefore that f is real. Also $\frac{1}{2}(|1 + f_1| + |1 - f_1|) = 1$ μ-a.e., and so we may assume that $-1 \leqq f \leqq 1$. Now we have

$$1 = \int_X |1 \pm f| \, d|\mu| = \int_X (1 \pm f) d|\mu| = \| \mu \| \pm |\mu|(f)$$

whence $|\mu|(f) = 0$. Q.E.D.

Lemma 2.2. *Let $\{\mu_\alpha\} \subset S_X$ be a net of positive measures, w^*-converging to $\mu \in S_X$ (where obviously $\mu \geqq 0$), and for each α let f_α be a Borel measurable real function such that $-1 \leqq f_\alpha \leqq 1$ and such that $f_\alpha \cdot \mu_\alpha \xrightarrow{w^*} \nu$. Then $\nu = f \cdot \mu$ for some Borel-measurable real f such that $-1 \leqq f \leqq 1$. If, in addition, $\mu_\alpha(f_\alpha) = 0$ for each α, then also $\mu(f) = 0$.*

Proof. By passing to subnets we may assume that

$$f_\alpha^+ \cdot \mu_\alpha \xrightarrow{w^*} \nu_1 \geqq 0$$

$$f_\alpha^- \cdot \mu_\alpha \xrightarrow{w^*} \nu_2 \geqq 0.$$

Hence

$$f_\alpha \cdot \mu_\alpha \xrightarrow{w^*} \nu_1 - \nu_2 = \nu = \nu^+ - \nu^-.$$

Therefore $\nu_1 \geqq \nu^+$ and $\nu_2 \geqq \nu^-$ and so $\nu_1 + \nu_2 \geqq |\nu|$. But

$$\nu_1 + \nu_2 = w^* - \lim |f_\alpha| \cdot \mu_\alpha \leqq w^* - \lim \mu_\alpha = \mu$$

(because $|f_\alpha| \leqq 1$). Hence $|\nu| \leqq \mu$ and the assertions follow immediately.

Q.E.D.

THEOREM 2.3. *Let X be a compact Hausdorff space with the property that for every countable $A \subset X$, $X - A$ contains at least two (relative) accumulation points. Let $F: S_X \to S_X$ be a continuous mapping such that $\|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in S_X$. Then, for each $\mu \in S_X$, there is a real Borel-measurable function f such that $-1 \leqq f \leqq 1$, $\big| |\mu|(f) \big| \leqq 1 - \|\mu\|$ and $F(\mu) = f \cdot \mu$.*

PROOF. It follows from Lemma 2.1 that $F(\mu)$ has this form if $\|\mu\| = 1$. Assume, therefore, that $\|\mu\| < 1$. Since μ has at most a countable number of atoms, it follows from the property of X that there exist at least two nets in X, $\{y_\alpha\}$, $\{z_\alpha\}$, converging to y, z respectively such that none of these points is an atom of μ. Take the first net and define, for each α, $\mu_\alpha = \frac{1}{2}\delta_{y_\alpha} - \frac{1}{2}\delta_y$ and $\nu_\alpha = \mu + (1 - \|\mu\|)\mu_\alpha$. Then $\|\mu_\alpha\| = 1$, μ_α is singular to μ and so $\|\nu_\alpha\| = 1$ for each α. Also $\nu_\alpha \xrightarrow{w^*} \mu$ and

$$|\nu_\alpha| = |\mu| + (1 - \|\mu\|)|\mu_\alpha| \xrightarrow{w^*} |\mu| + (1 - \|\mu\|)\delta_y.$$

Now, by Lemma 2.1, $F(\nu_\alpha) = f_\alpha \cdot \nu_\alpha$, where $-1 \leqq f \leqq 1$, are Borel-measurable functions such that $|\nu_\alpha|(f_\alpha) = 0$, for each α. Thus $f_\alpha \cdot \nu_\alpha \xrightarrow{w^*} F(\mu)$ and by taking a subnet if necessary, we may assume, by Lemma 2.2, that

$$f_\alpha \cdot |\nu_\alpha| \xrightarrow{w^*} f \cdot [|\mu| + (1 - \|\mu\|)\delta_y],$$

for some Borel–measurable real function f, $-1 \leqq f \leqq 1$ and $[|\mu| + (1 - \|\mu\|)\delta_y](f) = 0$. Hence $\big| |\mu|(f) \big| \leqq 1 - \|\mu\|$. (Actually, this last property is trivial once the rest of the theorem is proved.)

Now,

$$|\nu_\alpha \pm f_\alpha \cdot \nu_\alpha| = |\nu_\alpha| \pm f_\alpha \cdot |\nu_\alpha| \xrightarrow{w^*} (1 \pm f) \cdot [|\mu| + (1 - \|\mu\|\delta_y]$$

while $\nu_\alpha \pm f_\alpha \cdot \nu_\alpha \xrightarrow{w^*} \mu \pm F(\mu)$ and we have therefore

$$|\mu \pm F(\mu)| \leqq (1 \pm f) \cdot [|\mu| + (1 - \|\mu\|)\delta_y]$$

from which we have

$$|F(\mu)| \leqq \tfrac{1}{2}|\mu + F(\mu)| + \tfrac{1}{2}|\mu - F(\mu)| \leqq |\mu| + (1 - \|\mu\|)\delta_y.$$

Now repeat the whole procedure with the other net to obtain $|F(\mu)| \leqq |\mu|$. Hence there is a Borel-measurable function g such that $F(\mu) = g \cdot \mu$ and $|g| \leqq 1$. Thus,

$$|\mu \pm F(\mu)| = |1 \pm g| \cdot |\mu| \leqq (1 \pm f) \cdot [|\mu| + (1 - \|\mu\|)\delta_y].$$

But δ_y is singular to μ and so $|1 \pm g| \leqq 1 \pm f$ μ-a.e. and therefore $g = f$ μ-a.e. and this completes the proof. Q.E.D.

Remark 1. It is easy to check that every non-dispersed X (i.e., with a perfect non-void subset) satisfies the condition of the theorem. (Assume X to be perfect and $A \subset X$ to be countable. If $\bar{A} \neq X$ then all the points of $X - \bar{A}$ are relative accumulation points. If $\bar{A} = X$ then $X - A$ is a dense G_δ subset and each of its points is a relative accumulation point.) As we shall see later, this is the interesting case, for the dispersed spaces cannot provide counterexamples.

Remark 2. Since the theorems on extreme operators in the real case are based mainly on selection theorems (cf. [2]), it is interesting to note that for such X and real or complex scalars, the set-valued mapping $\Sigma: S_X \to 2^{S_X}$ defined by

$$\sum(\mu) = \{\nu \in S_X ; \|\mu \pm \nu\| \leqq 1\}, \quad \mu \in S_X$$

is not w^*-lower-semi-continuous, as follows easily from the theorem.

Theorem 2.4. *If the scalars are complex and X is a perfect compact Hausdorff space, then there does not exist a continuous mapping $F: S_X \to S_X$ such that $\|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in S_X$, except the identically zero mapping.*

Proof. It is sufficient to show that such F vanishes at every purely-atomic measure with finite number of atoms, and with norm less than 1. Let μ be such a measure, namely $\mu = \Sigma_{i=1}^n a_i\delta_{x_i}$ where, for each $1 \leqq i \leqq n$, $x_i \in X$, $a_i \neq 0$ and $\Sigma_{i=1}^n |a_i| < 1$. Now, let $\{p_i\}_{i=1}^n$ be a sequence of positive numbers such that for each i, $p_i > |a_i|$ and $\Sigma_{i=1}^n p_i = 1$. For each $1 \leqq i \leqq n$ there are two complex numbers Z_i, W_i such that

$$a_i = Z_i + W_i, \qquad p_i = |Z_i| + |W_i|$$

and Z_i, W_i are linearly independent over the reals. Since X is perfect, there is, for each i, a net $\{y_i^\alpha\}$ of different points in X which converges to x_i. Consider the following net of measures:

$$\mu_\alpha = \sum_{i=1}^{n} [Z_i\delta_{x_i} + W_i\delta_{y_i^\alpha}].$$

We may assume that for each α, all the points, x_i, y_i^α, $1 \leqq i \leqq n$, are different from each other. Thus

$$|\mu_\alpha| = \sum_{i=1}^{n} (|Z_i|\,\delta_{x_i} + |W_i|\,\delta_{y_i^\alpha}) \quad \text{and} \quad \|\mu_\alpha\| = 1 \quad \text{for each} \quad \alpha.$$

We have also $\mu_\alpha \xrightarrow{w^*} \mu$ and $|\mu_\alpha| \xrightarrow{w^*} \Sigma_{i=1}^n p_i\delta_{x_i}$.

Now, by Lemma 2.1, $F(\mu_\alpha) = f_\alpha \cdot \mu_\alpha$ for each α, where $-1 \leqq f_\alpha \leqq 1$ is Borel-measurable and $|\mu_\alpha|(f_\alpha) = 0$. That is

$$\sum_{i=1}^{n} (|Z_i|\,f_\alpha(x_i) + |W_i|\,f_\alpha(y_i^\alpha)) = 0.$$

By taking subnets if necessary we may assume that

$$\lim_\alpha f_\alpha(x_i) = c_i; \quad \lim_\alpha f_\alpha(y_i^\alpha) = d_i, \quad i = 1,2,\cdots,n.$$

Thus $\Sigma_{i=1}^n(|Z_i|\,c_i + |W_i|\,d_i) = 0$.
We also have

$$F(\mu_\alpha) = f_\alpha \cdot \mu_\alpha = \sum_{i=1}^{n} [Z_i f_\alpha(x_i)\delta_{x_i} + W_i f_\alpha(y_i^\alpha)\delta_{y_i^\alpha}] \xrightarrow{w^*} \sum_{i=1}^{n} (Z_i c_i + W_i d_i)\delta_{x_i}.$$

But F is continuous and so $F(\mu_\alpha) \xrightarrow{w^*} F(\mu)$, and by Theorem 2.3, there is a (measurable) function f such that $-1 \leqq f \leqq 1$ and

$$F(\mu) = f \cdot \mu = \sum_{i=1}^{n} f(x_i)(Z_i + W_i)\delta_{x_i}.$$

We conclude, therefore, that

$$Z_i c_i + W_i d_i = (Z_i + W_i) f(x_i) \qquad i = 1,2,\cdots,n,$$

and from the linear independence of Z_i, W_i we have for each i, $c_i = d_i = f(x_i)$. Hence

$$0 = \sum_{i=1}^{n} (|Z_i| c_i + |W_i| d_i) = \sum_{i=1}^{n} p_i f(x_i).$$

But $\{p_i\}$ was almost arbitrary, and it is easily verified that the last equality can hold for every suitable sequence $\{p_i\}$ only if $f(x_i) = 0$ for each $1 \leqq i \leqq n$. Hence $F(\mu) = 0$ and it follows that F is identically zero. Q.E.D.

THEOREM 2.5. *If the scalars are complex and X is a perfect compact Hausdorff space, then the operator $T: C(X) \to C(S_X)$, defined by*

$$Tf(\mu) = \mu(f), \quad f \in C(X), \quad \mu \in S_X,$$

is an extreme operator in the unit ball of $L(C(X), C(S_X))$ but is not a nice operator.

PROOF. Consider the corresponding w^*-continuous mapping $T^*: S_X \to S_X$ which is actually the identity on S_X, namely $T^*\mu = \mu$ for each $\mu \in S_X$. Obviously T is not nice. Nevertheless it is extreme, for otherwise there would exist a w^*-continuous mapping $U^*: S_X \to S_X$ which does not vanish identically and such that $\|T^*\mu \pm U^*\mu\| \leqq 1$ for each $\mu \in S_X$, which contradicts Theorem 2.4. Q.E.D.

REMARK 1. Note that T is an isometry into. We also have by [10, lemma 2.5] that T^* is not extreme. Another property of T is that if J is the canonical imbedding of $C(S_X)$ into $C(S_X)^{**}$, then JT is not extreme [9]. Hence the composition of two extreme operators between complex C-spaces, each of which is an isometry into, need not be extreme.

REMARK 2. The requirement that X is perfect is essential to Theorem 2.4. Indeed, if $x_0 \in X$ is isolated, then the mapping $F(\mu) = \frac{1}{2}\mu(\{x_0\})(\mu - \delta_{x_0})$ $\mu \in S_X$, isn't identically zero, and $\|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in S_X$. Nevertheless, the following proposition holds in the more general case.

COROLLARY 2.6. *Let X be a non-dispersed compact Hausdorff space. If the scalars are complex, then there exist a compact Hausdorff space Y and an extreme operator T in the unit ball of $L(C(X), C(Y))$ which is not nice.*

PROOF. Let $A \subset X$ be a non-void perfect subset, and let Y be the subset of S_X containing those measures whose support is contained in A. Y is obviously w^*-compact. Define $T: C(X) \to C(Y)$ by the adjoint mapping which imbeds Y in S_X, i.e. $T^*\mu = \mu$, $\mu \in Y$. T is clearly not nice, but it is extreme, for if $F: Y \to S_X$ is w^*-continuous and $\|T^*\mu \pm F(\mu)\| = \|\mu \pm F(\mu)\| \leqq 1$ for each $\mu \in Y$, then $F(Y) \subset Y$ (this is true for norm-one measures by Lemma 2.1, and these measures are dense in Y). Now it follows from Theorem 2.4 that $F \equiv 0$, for Y is w^*-affinely homeomorphic to S_A and A is perfect. Thus T is extreme. Q.E.D.

COROLLARY 2.7. *In the complex case, the dispersed compact Hausdorff spaces are the only spaces X with the property that for every compact Hausdorff space Y, every extreme operator in $L(C(X), C(Y))$ is nice.*

PROOF. Immediate, since it was shown in [9] that every dispersed X has this property.

We can enlarge the class of extreme non-nice operators in the following way:

THEOREM 2.8. *Let the scalars be complex, and let X be a perfect metric compact space. If Y is a compact Hausdorff space such that there exists an* open *continuous mapping φ of Y onto S_X, then the operator $T: C(X) \to C(Y)$ defined by $T^*y = \varphi(y)$, $y \in Y$, is extreme but not nice.*

PROOF. That T is not nice is obvious. If T is not extreme, then there exists a continuous mapping $\psi: Y \to S_X$ which does not vanish identically and such that $\| \varphi(y) \pm \psi(y) \| \leqq 1$ for each $y \in Y$. Define now the set-valued mapping $\Sigma: S_X \to 2^{S_X}$ by $\Sigma(\mu) = \overline{\mathrm{conv}}^{w^*} \psi[\varphi^{-1}(\{\mu\})]$, $\mu \in S_X$. For each $\mu \in S_X$, $\Sigma(\mu)$ is a non-void compact convex subset of S_X. We shall show that Σ is w^*-lower semi-continuous in the sense of Michael [7]. It is sufficient to show that for each $\mu_0 \in S_X$, $\nu_0 \in \psi[\varphi^{-1}(\{\mu_0\})]$ and V_0 a w^*-neighborhood of ν_0, there cannot exist a net $\{\mu_\alpha\}$ of measures in S_X such that $\mu_\alpha \xrightarrow{w^*} \mu_0$ and $\Sigma(\mu_\alpha) \cap V_0 = \varnothing$ for each α. Indeed, let $\{\mu_\alpha\}$ be such a net and write $\nu_0 = \psi(y_0)$ where $y_0 \in Y$ and $\varphi(y_0) = \mu_0$. Since φ is open, there exists a net $\{y_\alpha\} \subset Y$ such that $y_\alpha \to y_0$ and $\varphi(y_\alpha) = \mu_\alpha$. Then we must have eventually $\psi(y_\alpha) \in \Sigma(\mu_\alpha) \cap V_0$. This contradiction shows that Σ is lower semi-continuous. Now, S_X is w^*-metrizable, and so, by Michael's selection theorem [7] there exists a continuous mapping $F: S_X \to S_X$ such that $F(\mu) \in \Sigma(\mu)$ for each $\mu \in S_X$ and F doesn't vanish identically. But then $\| \mu \pm F(\mu) \| \leqq 1$ for each $\mu \in S_X$. This contradicts Theorem 2.4 and therefore T must be extreme. Q.E.D.

REMARK 1. If φ is just a continuous surjection of Y onto S_X, the theorem may fail, as might be seen by taking $Y = \beta N$ and define φ by mapping N onto a dense countable subset of S_X. The resulting operator is not extreme because it is not nice, and Y has a dense set of isolated points (cf. [2]).

REMARK 2. Similar results to Theorem 2.8 can be stated for non-dispersed X, in view of Corollary 2.6.

REMARK 3. If X is a compact metric space, then S_X is easily seen to be homeomorphic to a norm-compact convex subset of l^2, and therefore, by Keller's theorem [6], homeomorphic to the Hilbert-cube. Thus, in the metric case, our counterexample is actually an extreme non-nice operator in $L(C(X), C([0,1]^{\aleph_0}))$. It is an open problem, whether $[0,1]^{\aleph_0}$ may be replaced by "smaller" metric spaces. For example: Does there exist an extreme non-nice operator in $L(C[0,1], C[0,1])$?

REFERENCES

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. Math. **88** (1968), 35–46.

2. R. M. Blumenthal, J. Lindenstrauss and R. R. Phelps, *Extreme operators into $C(K)$*, Pacific J. Math. **15** (1965), 747–756.

3. H. H. Corson and J. Lindenstrauss, *Continuous selections with non-metrizable range*, Trans. Amer. Math. Soc. **121** (1966). 492–504.

4. N. Dunford and J. Schwartz, *Linear Operators*, Vol. I, Interscience, New York, 1958.

5. O. Hustad, *Intersection properties of balls in complex Banach spaces whose duals are L^1-spaces*, Acta Math. **132** (1974), 283–313.

6. O. H. Keller, *Die Homoisomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

7. E. Michael, *A linear mapping between function spaces*, Proc. Amer. Math.. Soc. **15** (1964), 407–409.

8. P. D. Morris and R. R. Phelps, *Theorems of Krein-Milman type for certain convex sets of operators*, Trans. Amer. Math. Soc. **150** (1970), 183–200.

9. M. Sharir, *Characterization and properties of extreme operators into $C(Y)$*, Israel J. Math. **12** (1972), 174–183.

10. M. Sharir, *Extremal structure in operator spaces*, Trans. Amer, Math. Soc. **186** (1973), 91–111.

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